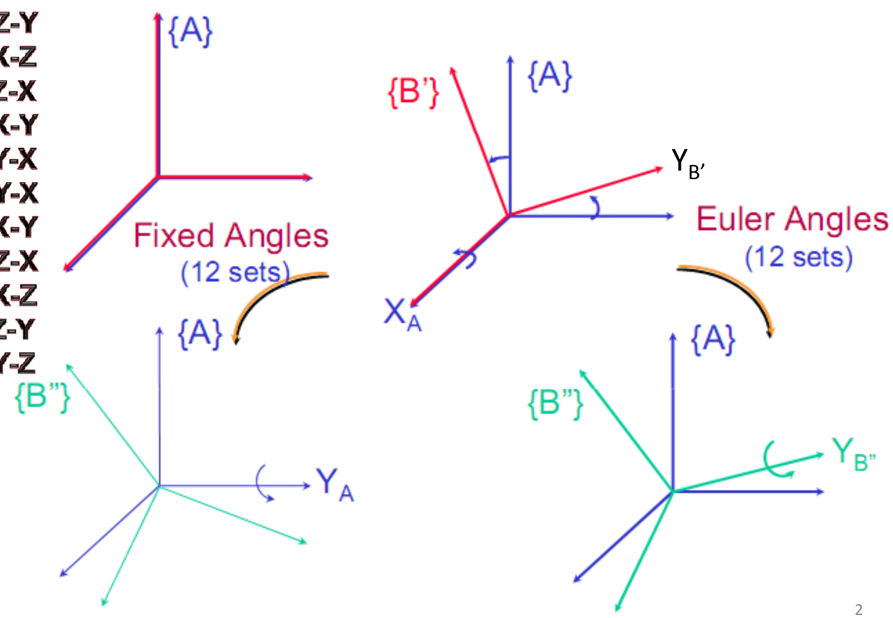
Lecture 4

Orientation Representation

The generic problem of attitude representation

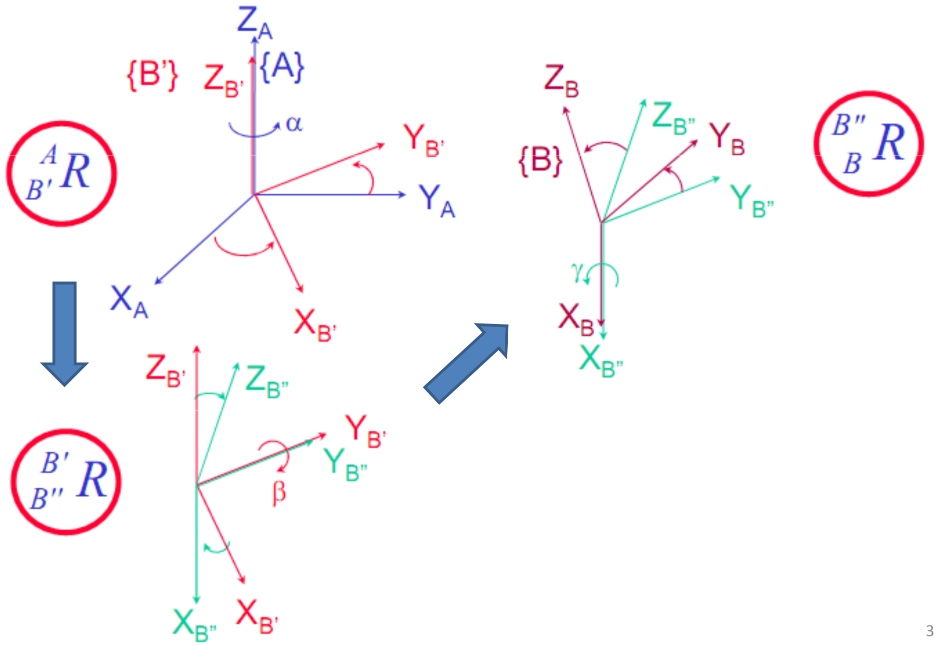
Three Angle Representations

1 X-Y-Z
2 X-Z-Y
3 Y-X-Z
4 Y-Z-X
5 Z-X-Y
6 Z-Y-X
7 X-Y-X
8 Y-X-Y
9 X-Z-X
10 Z-X-Z
11 Y-Z-Y
12 Z-Y-Z



Euler Angles (Z-Y-X)

6th one on the list



Euler Angles: Successive rotations are around the axes of the moving frame

$$^{B''}P = {}^{B''}_{B}R\,{}^{B}P$$

$$^{B'}P = {}^{B'}_{B''}R\,{}^{B''}P = {}^{B'}_{B''}R\,{}^{B''}_{B}R\,{}^{B}P$$

$$^{A}P = {}^{A}_{B'}R\,{}^{B'}P = {}^{A}_{B'}R\,{}^{B'}_{B''}R\,{}^{B''}_{B}R\,{}^{B}P \quad (1)$$

But

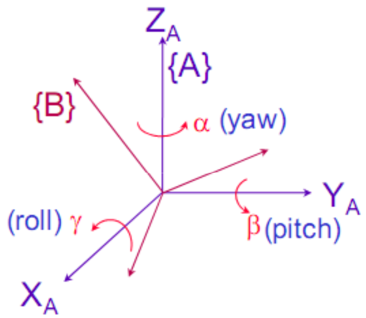
$$^{A}P = {}^{A}_{B}R\,{}^{B}P \quad (2)$$

First rotation | Second rotation | Third rotation

From (1) and (2), $${}^{A}_{B}R_{Euler} = {}^{A}_{B'}R\,{}^{B'}_{B''}R\,{}^{B''}_{B}R$$

$$R_z(\alpha)R_Y(\beta)R_X(\gamma)$$

1st one on the list ➡ X-Y-Z Fixed Angles



After first rotation $^{A}P \Rightarrow R_X(\gamma)^{B}P$

After second rotation $^{A}P \Rightarrow R_Y(\beta)^{A}P = R_Y(\beta)R_X(\gamma)^{B}P$

After third rotation $^{A}P \Rightarrow R_Z(\alpha)^{A}P = R_Z(\alpha)R_Y(\beta)R_X(\gamma)^{B}P$ (3)

But

$^{A}P = {}^{A}_{B}R\,^{B}P$ (4)

Third rotation, Second rotation, First rotation

From (3) and (4), ${}^{A}_{B}R_{fixed} = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$

Euler Angle Rotation $R_Z(\alpha)R_Y(\beta)R_X(\gamma)$

$$\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

Basic rotation matrices

$$\begin{bmatrix} c\alpha.c\beta & c\alpha.s\beta.s\gamma - s\alpha.c\gamma & c\alpha.s\beta.c\gamma + s\alpha.s\gamma \\ s\alpha.c\beta & s\alpha.s\beta.s\gamma + c\alpha.c\gamma & s\alpha.s\beta.c\gamma - c\alpha.s\gamma \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix}$$

Inverse Problem

- **Determine α β γ from the rotation matrix**

$${}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha.c\beta & c\alpha.s\beta.s\gamma - s\alpha.c\gamma & c\alpha.s\beta.c\gamma + s\alpha.s\gamma \\ s\alpha.c\beta & s\alpha.s\beta.s\gamma + c\alpha.c\gamma & s\alpha.s\beta.c\gamma - c\alpha.s\gamma \\ -s\beta & c\beta.s\gamma & c\beta.c\gamma \end{bmatrix}$$

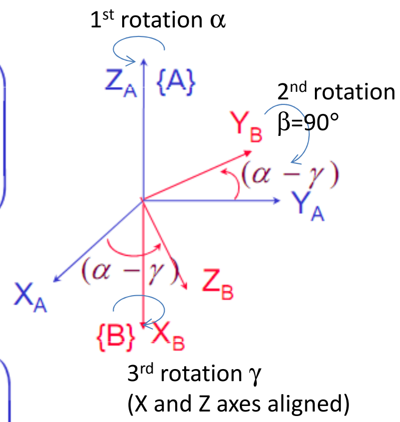
$$\left.\begin{aligned} \cos\beta = c\beta = \sqrt{r_{11}^2 + r_{21}^2} \\ \sin\beta = s\beta = -r_{31} \end{aligned}\right\} \rightarrow \beta = A\tan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

if $c\beta = 0$ ($\beta = \pm 90°$) ⇨ Singularity of the representation

Singular Configurations (β=90°)

$c\beta = 0,\ s\beta = +1$

$${}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{pmatrix}$$



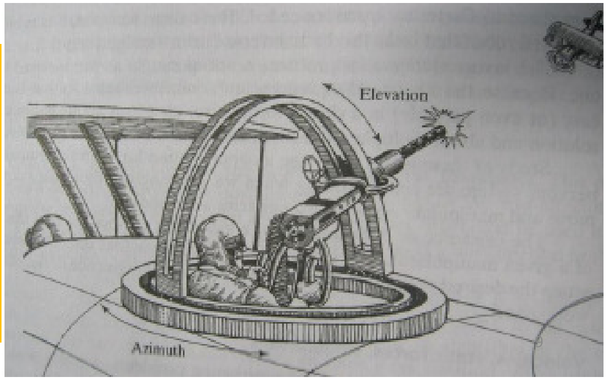
$c\beta = 0,\ s\beta = -1$

$${}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha+\gamma) & -c(\alpha+\gamma) \\ 0 & c(\alpha+\gamma) & -s(\alpha+\gamma) \\ 1 & 0 & 0 \end{pmatrix}$$

Singular Configurations

- World War 1 Vintage biplane : the gunner controls azimuth and elevation (2dof) to target enemy planes
- **Guiding question**: At which of the two elevations 25º and 80º, is targeting enemy plane easier?
- **Guiding question**: What about the targeting capability of the mechanism when the enemy plane is at 90º elevation? How many effective dofs are there at 90º elevation?

A mechanism is said to be locally degenerated when it behaves as if it has lost a one or more dofs, which happens at at some orientations, known as singularities



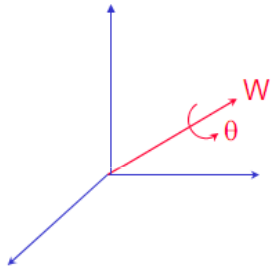
Euler Parameters

$$\varepsilon_1 = W_x . \sin\frac{\theta}{2}$$

$$\varepsilon_2 = W_y . \sin\frac{\theta}{2}$$

$$\varepsilon_3 = W_z . \sin\frac{\theta}{2}$$

$$\varepsilon_4 = \cos\frac{\theta}{2}$$

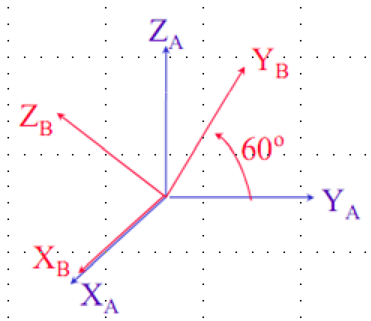


Normality Condition

$$|W| = 1, \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$$

ε : point on a unit hypershpere in four-dimensional space

Q. Determine the Euler parameters for the following rotation



Direction Cosine Representation

$${}^A_B R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.87 \\ 0 & 0.87 & 0.5 \end{pmatrix}$$

Euler Parameter Representation

$${}^A_B \mathrm{E} = \begin{pmatrix} 0.5 \\ 0 \\ 0 \\ 0.87 \end{pmatrix}$$

Euler Parameters / Euler Angles

$$\varepsilon_1 = \sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2}$$

$$\varepsilon_2 = \sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}$$

$$\varepsilon_3 = \cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}$$

$$\varepsilon_4 = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}$$

Inverse Problem

Given ${}^{A}_{B}R$ find ε

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \equiv \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}$$

$$r_{11}+r_{22}+r_{33} = 3-4(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2)$$

$$(1-\varepsilon_4^2)$$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$

$$\varepsilon_1 = \frac{r_{32}-r_{23}}{4\varepsilon_4}, \quad \varepsilon_2 = \frac{r_{13}-r_{31}}{4\varepsilon_4}, \quad \varepsilon_3 = \frac{r_{21}-r_{12}}{4\varepsilon_4}$$

$\varepsilon_4 = 0$?

Lemma

For all rotations one of the Euler Parameters is greater than or equal to 1/2

$$\left(\sum_{1}^{4} \varepsilon_i^2 = 1\right)$$

Algorithm

Solve with respect to $\max_i\{\varepsilon_i\}$

- $\varepsilon_1 = \max_i\{\varepsilon_i\}$

$$\varepsilon_1 = \frac{1}{2}\sqrt{r_{11}-r_{22}-r_{33}+1}$$

$$\varepsilon_2 = \frac{(r_{21}+r_{12})}{4\varepsilon_1}, \quad \varepsilon_3 = \frac{(r_{31}+r_{13})}{4\varepsilon_1}, \quad \varepsilon_4 = \frac{(r_{32}-r_{23})}{4\varepsilon_1}$$

- $\varepsilon_2 = \max_i\{\varepsilon_i\}$

$$\varepsilon_2 = \frac{1}{2}\sqrt{-r_{11}+r_{22}-r_{33}+1}$$

- $\varepsilon_3 = \max_i\{\varepsilon_i\}$

$$\varepsilon_3 = \frac{1}{2}\sqrt{-r_{11}-r_{22}+r_{33}+1}$$

- $\varepsilon_4 = \max_i\{\varepsilon_i\}$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$